Estimates for Conformal Capacity

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Abstract. Let a, b, c, d be distinct points on $\overline{\mathbf{R}}^n$. By p we denote the minimal conformal capacity of all rings (E, F) with $a, b \in E$ and $c, d \in F$. For $n = 2$, we use explicit expressions of p in terms of complete elliptic integrals to prove a sharp inequality that connects p and the conformal capacity of Teichmüller's ring. We also show, by a concrete example, how we can use techniques involving polarization and hyperbolic geometry to prove estimates for the conformal capacity of rings.

1. Introduction

A *condenser* in $\overline{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$, $n \geq 2$, is a pair (E, F) , where E, F are nonempty, disjoint, compact subsets of $\overline{\mathbf{R}}^n$. If both E and F are connected, then the condenser (E, F) is called a *ring*. The *conformal capacity* of a condenser (E, F) with $\infty \in F$ is defined as

$$(1.1) \quad \text{cap}(E, F) = \inf_u \int_R |\nabla u|^n dx,$$

where $R = \overline{\mathbf{R}}^n \setminus (E \cup F)$ and the infimum is taken over all nonnegative $C^\infty(\mathbf{R}^n \setminus F)$ functions with compact support in $\mathbf{R}^n \setminus F$ such that $u(x) \geq 1$, for $x \in E$ (see [AVV, p. 165]). A theorem of F. W. Gehring [G1] says that the conformal capacity of a condenser (E, F) is equal to the module of the family of curves joining E and F . In particular, the conformal capacity is conformally invariant. Using conformal invariance we define the conformal capacity of arbitrary condensers (see [V1, p. 84] for details).

For distinct points $a, b, c, d \in \overline{\mathbf{R}}^n$, let

$$(1.2) \quad p(a, b; c, d) = \inf_{E, F} \text{cap}(E, F),$$

where the infimum is taken over all rings (E, F) with $a, b \in E$ and $c, d \in F$. O. Teichmüller [Te] posed (for $n = 2$) the problem of finding the value of $p(a, b; c, d)$ for each choice of a, b, c, d . By conformal invariance we may assume $a = O$, $b = e_1$,

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$c = x \in \mathbf{R}^n \setminus \{O, e_1\}$, and $d = \infty$. Here O denotes the origin in $\mathbf{R}^n$ and $e_1, \dots, e_n$ are the standard unit vectors in $\mathbf{R}^n$. We set

$$p(x) = p(O, e_1; x, \infty).$$

This is *Teichmüller's function*. M. Schiffer showed that in the two-dimensional case the extremal continua E, F are images of linear sets under an elliptic function. Further results, historic remarks, and references appear in G. V. Kuz'mina's book [Ku, Chap. 5]. For $n = 2$, the function p can be expressed in terms of elliptic integrals with complex argument [Ku, p. 192]. Another way to study p in all dimensions is to find upper and lower bounds. Indeed, spherical symmetrization yields the following lower bound [AVV, p. 304]:

$$(1.3) \quad p(x) \geq \tau_n(\min\{|x|, |x - e_1|\}),$$

where τ_n denotes the conformal capacity of Teichmüller's ring (see Section 2). This inequality is sharp and equality holds for $x = se_1$ with $s \in (-\infty, 0) \cup (1, \infty)$. Nevertheless, A. Yu. Solynin and M. Vuorinen [SV] used a duplication formula for p to show that for $n = 2$, the lower bound (1.3) can be improved in many cases. We review lower bounds for Teichmüller's function in Section 2. Vuorinen [V2] found an upper bound for $p(x)$ when $x \in \mathbf{R}^n \setminus [O, e_1]$:

$$(1.4) \quad p(x) \leq \tau_n \left(\frac{|x| + |x - e_1| - 1}{2} \right), \quad x \in \mathbf{R}^n \setminus [O, e_1].$$

In general (i.e., for every $x \in \mathbf{R}^n \setminus \{O, e_1\}$), we have [Be]:

$$(1.5) \quad p(x) \leq 4\tau_n(\min\{|x|, |x - e_1|\}).$$

For $n = 2$ this bound can be improved. The next theorem, which is the main result of the present paper, gives an upper bound for $p(x)$ in the two-dimensional case.

Theorem 1.1. *Let $x \in \mathbf{R}^2 \setminus \{O, e_1\}$. Then*

$$(1.6) \quad p(x) \leq \tau_2(2)\tau_2(\min\{|x|, |x - e_1|\}).$$

Equality holds if and only if $x = e_1/2$.

This theorem is complementary to the lower bound (1.3). The proof is given in Section 3. It involves explicit expressions of the functions p and τ_2 in terms of elliptic integrals. The constant $\tau_2(2) = 4/\tau_2(\frac{1}{2})$ is approximately equal to 1.71. Note that for $n = 2$, $p(e_1/2) = 4$. This follows from an explicit expression for the function p (see (3.5)). The inequality (1.6) is of some importance because the explicit expression of p involves the little studied complete elliptic integrals of complex argument. Some other upper bounds for $p(x)$ in the two-dimensional case appear in [LV].

In Section 4 we prove upper bounds for the conformal capacity of rings (K, L_θ) where $K = [O, e_1]$, $L_\theta = \{se_1 : s \geq 1\}$, $e = (\cos \theta)e_1 + (\sin \theta)e_n$, and $\theta \in (0, \pi]$. For $n = 2$, we use a technique that involves the interpretation of conformal capacity as Green capacity. In higher dimensions such a technique does not apply and we rely on

polarization. The bounds for $n = 2$ are much better than those in higher dimensions. In the planar case the conformal capacity $\text{cap}(K, L_\theta)$ can be computed by a recently found numerical method [Hu] based on the Schwarz–Christoffel transform for doubly connected polygonal domains. In Section 4 we present a table which contains the values of our bounds, the numerical approximations of the actual values of $\text{cap}(K, L_\theta)$, and lower bounds obtained via the method of Solynin and Vuorinen mentioned above.

2. Background Material

2.1. Teichmüller's Ring and Complete Elliptic Integrals

For $x, y \in \mathbf{R}^n$, let $[x, y] = \{(1-t)x + ty : 0 \leq t \leq 1\}$. If $x \neq O$ we set $[x, \infty] = \{tx : t \geq 1\} \cup \{\infty\}$. The ring $([-e_1, O], [te_1, \infty])$, where $t > 0$, is called *Teichmüller's ring*. Its capacity is denoted by $\tau_n(t)$. For $n = 2$, τ_n (also denoted by τ) has an explicit expression in terms of elliptic integrals [AVV, Chap. 8]:

$$(2.1) \quad \tau_2(t) = \tau(t) = \frac{\pi}{\mu(1/\sqrt{1+t})},$$

where

$$\mu(r) = \frac{\pi}{2} \frac{\mathcal{K}'(r)}{\mathcal{K}(r)}, \quad r \in (0, 1).$$

We adopted above the standard notation for the complete elliptic integrals of the first kind [AVV, Chap. 3]:

$$\mathcal{K}(r) = \int_0^1 \frac{dt}{(1-t^2)^{1/2}(1-r^2t^2)^{1/2}}, \quad \mathcal{K}'(r) = \mathcal{K}(r'), \quad r' = \sqrt{1-r^2}, \quad r \in (0, 1).$$

The function τ satisfies the following duplication formula [AVV, p. 89]:

$$(2.2) \quad \tau\left(\frac{1}{e^{2t}-1}\right) = 2\tau\left(\frac{2}{\cosh t - 1}\right), \quad t > 0.$$

The derivative of the function μ has an explicit expression [AVV, p. 82]:

$$(2.3) \quad \frac{d\mu}{dr} = \frac{-\pi^2}{4r(r')^2\mathcal{K}(r)^2}.$$

For $n \geq 3$, there are no known expressions for the function τ_n and so we rely on qualitative properties and inequalities: τ_n is a strictly decreasing function of t and satisfies the following inequality [AVV, Theorem 11.25]:

$$(2.4) \quad \tau_n(ct) \leq \frac{1}{\sqrt{c}} \tau_n(t), \quad 0 < c \leq 1, \quad t > 0.$$

The ring $(\{x : |x| \leq 1\}, [te_1, \infty])$, where $t > 1$, is called *Grötzsch's ring*. It plays an important role in various branches of analysis (see, e.g., [AVV, Chaps. 7, 8]). For $n = 2$ its

conformal capacity will be denoted by $\gamma(t)$. Grötzsch's ring can be mapped conformally onto Teichmüller's ring. The conformal invariance of capacity implies [AVV, p. 88]:

$$(2.5) \quad \gamma(t) = \tau \left(\frac{(t-1)^2}{4t} \right), \quad t > 1.$$

2.1. Lower Bounds for Teichmüller's Function in Plane

Because of symmetry, to find lower bounds for $p(z)$, $z \in \mathbb{C}$ we may assume that $z \in I := \{z: \Re z \geq 1/2, \Im z \geq 0\} \setminus \{1\}$. Solynin and Vuorinen [SV] discovered the following duplication formula:

$$(2.6) \quad p(z) = 2p(w^4), \quad w = \sqrt{z} + \sqrt{z-1}, \quad z \in I.$$

The branches of the square roots are chosen so that $\arg \sqrt{z} \in [0, \pi/2]$ and $\arg \sqrt{z-1} \in [0, \pi/2]$ for $z \in I$. The duplication formula (2.6) leads to lower bounds for p that improve (1.3):

$$(2.7) \quad p(z) = 2p(w^4) \geq 2\tau(\min\{|w|^4, |w^4 - 1|\}), \quad z \in I.$$

Repeated applications of the duplication formula (2.6) and the symmetrization inequality (1.3) may further improve the lower bound obtained by the direct application of (1.3).

To facilitate the usage of (2.7), we express $|w|^4$ and $|w^4 - 1|$ in terms of geometric quantities involving z only. Elementary calculations show that

$$(2.8) \quad |w|^4 = (r_0 + r_1 + \sqrt{(r_0 + r_1)^2 - 1})^2,$$

$$(2.9) \quad |w^4 - 1| = 4\sqrt{r_0 r_1}(r_0 + r_1 + \sqrt{(r_0 + r_1)^2 - 1}),$$

where $r_0 = |z|$, $r_1 = |z - 1|$, and $z \in I$. If $z \notin I$ then (by symmetry) there exists $z' \in I$ such that $p(z) = p(z')$. Then we can apply (2.7) with $w = \sqrt{z'} + \sqrt{z'-1}$.

In Section 4 we find some upper bounds for $p(z)$ and compare them with the lower bounds obtained by the method presented above.

2.2. Green Transfinite Diameter

Let D be a simply connected domain in the extended complex plane $\widehat{\mathbb{C}}$ and assume that the boundary ∂D of D has positive logarithmic capacity. Let L be a continuum in D . The *Green capacity* $\text{cap}_D(L)$ of L with respect to D is defined by the standard potential theoretic method that involves the minimization of the energy integral (see, e.g., [La, p. 174]). Let

$$E_D(L) = \min \int_L \int_L g_D(z_1, z_2) d\nu(z_1) d\nu(z_2),$$

where the minimum is taken over all unit Borel measures ν on L , and g_D is the Green function of D . Then

$$\text{cap}_D(L) = \frac{1}{E_D(L)}.$$

The Green capacity $\text{cap}_D(L)$ is equal to the *Green transfinite diameter* $T_D(L)$ of L with respect to D (see [La, p. 175]). $T_D(L)$ is defined as follows: For $z_1, \dots, z_m \in L$, let

$$V(z_1, \dots, z_m) = \frac{2}{m(m-1)} \sum_{i < j} g_D(z_i, z_j), \quad V_m(L) = \max_{z_j \in L} V(z_1, \dots, z_m).$$

Then

$$T_D(L) = \lim_{m \rightarrow \infty} \exp(-V_m(L)).$$

This definition is similar to the definition of the hyperbolic transfinite diameter studied by M. D. Tsuji [Ts, pp. 94–96]. The equality $T_D(L) = \text{cap}_D(L)$ is essentially due to Tsuji.

For our purposes the following equality is important (see [La, p. 97]):

$$(2.10) \quad \frac{1}{2\pi} \text{cap}(\partial D, L) = \text{cap}_D(L) = T_D(L).$$

Thus the Green function g_D becomes a useful tool for the estimation of the conformal capacity of planar rings.

2.3. Green Function and Hyperbolic Metric

Let D be a simply connected domain in $\widehat{\mathbf{C}}$ and assume that ∂D has positive logarithmic capacity. Let $\lambda_D(z)$ denote the hyperbolic density of D at $z \in D$ and let $\rho_D(z_1, z_2)$ denote the hyperbolic distance of $z_1, z_2 \in D$. For the definition and main properties of λ_D and ρ_D we refer to [Ha, Sec. 9.4]. The Green function g_D is connected with ρ_D via the following formula:

$$(2.11) \quad g_D(z_1, z_2) = \log \frac{1 + e^{-2\rho_D(z_1, z_2)}}{1 - e^{-2\rho_D(z_1, z_2)}}.$$

To prove (2.11), by conformal invariance we may assume that $D = \mathbf{D} = \{z : |z| < 1\}$, $z_1 = 0$, and $z_2 \in \mathbf{R}$. Then (2.11) follows from explicit formulas for $g_{\mathbf{D}}$ and $\rho_{\mathbf{D}}$.

The identity (2.11) implies that for $z_1, z_2 \in D$, the inequality $g_D(\tilde{z}_1, \tilde{z}_2) \geq g_D(z_1, z_2)$ is equivalent to the inequality $\rho(\tilde{z}_1, \tilde{z}_2) \leq \rho(z_1, z_2)$. Therefore we can use the hyperbolic geometry of D for the estimation of $\text{cap}(\partial D, L)$, where L is a continuum in D . The use of $\rho_D(z_1, z_2)$ for this purpose, compared with the use of $g_D(z_1, z_2)$, has the advantage that ρ_D is given as an integral of λ_D . The study of λ_D can exploit efficiently the local geometry of D .

2.4. Polarization

Polarization is a geometric transformation in $\overline{\mathbf{R}}^n$. For $x = (x_1, \dots, x_n) \in \mathbf{R}^n$, let $x^* = (x_1, \dots, x_{n-1}, -x_n)$. Let also $\infty^* = \infty$. If A is a compact subset of $\overline{\mathbf{R}}^n$, let $A^* = \{x^* : x \in A\}$, $A^+ = \{x \in A : x_n \geq 0\}$, $A^- = \{x \in A : x_n \leq 0\}$. The *polarization* $\mathbf{P}A$ of A with respect to the oriented $(n-1)$ -dimensional plane $\{x_n = 0\}$ is defined by

$$(2.12) \quad \mathbf{P}A = (A \cup A^*)^+ \cup (A \cap A^*)^-.$$

If (A, B) is a condenser in $\overline{\mathbf{R}}^n$ the polarization $\mathbf{P}(A, B)$ of (A, B) with respect to $\{x_n = 0\}$ is the condenser

$$(2.13) \quad \mathbf{P}(A, B) = (\mathbf{P}A, (\mathbf{P}B)^*).$$

Now, let H be an oriented $(n-1)$ -dimensional plane or sphere in $\mathbf{R}^n$. Let T be a Möbius transformation that maps $\{x_n = 0\}$ (with its natural orientation) onto H , preserving their orientation. Then we define the polarization $\mathbf{P}_H A$ of a compact set $A \subset \overline{\mathbf{R}}^n$ by

$$(2.14) \quad \mathbf{P}_H A = T\mathbf{P}T^{-1}A$$

and similarly we define the polarization $\mathbf{P}_H(A, B)$ of the condenser (A, B) with respect to H .

The behavior of conformal capacity under polarization has been studied by V. Wolontis [Wo] for $n = 2$. Later V. N. Dubinin renewed the interest to polarization, introduced the name “polarization,” and (together with others) established its importance in geometric function theory (see [D1] and [D2] and references therein). The main polarization result for conformal capacity is the following:

Theorem 2.1. *If (A, B) is a condenser in $\mathbf{R}^n$ and H is an oriented $(n-1)$ -dimensional plane or sphere, then*

$$(2.15) \quad \text{cap}(A, B) \geq \text{cap } \mathbf{P}_H(A, B).$$

We used above the terminology of $\mathbf{R}^n$, $n \geq 3$. For $n = 2$, we must replace the expression “ $(n-1)$ -dimensional plane or sphere” by the expression “straight line or circle.”

3. Proof of the Main Theorem

In this section we prove Theorem 1.1. We identify $\mathbf{R}^2$ with $\mathbf{C}$ and use complex notation. We start with some reductions of the problem. Let $z \in \mathbf{C} \setminus \{0, 1\}$. Because of symmetry we may assume that $|1 - z| \leq |z|$ and $\Im z \geq 0$.

The inequality (1.4) implies (see [AVV, p. 304]):

$$(3.1) \quad p(z) \leq \sqrt{2} \tau(|1 - z|), \quad |z| > 1.$$

Since $\sqrt{2} < \tau(2)$, we may further assume that $|z| \leq 1$.

For $r \in (0, \infty)$, $p(1 + re^{i\varphi})$ is a strictly decreasing function of $\varphi \in (0, \pi)$. This is a result of Kuz'mina [Ku, p. 206], see also Lemma 4.2 in [Be] for a generalization in higher dimensions. It shows that we can further assume that either $z = \frac{1}{2} + iy$, $y \in [0, 1]$, or $z = x \in [\frac{1}{2}, 1)$.

Let $y \in [0.3, 1]$, $z = \frac{1}{2} + iy$, $l(y) = (y^2 + \frac{1}{4})^{1/2} = |1 - z|$, and $l_0 = (0.3^2 + \frac{1}{4})^{1/2}$. Then by (1.4), and the fact that τ is a decreasing function

$$(3.2) \quad p(z) = p(\tfrac{1}{2} + iy) \leq \tau\left(\frac{l+l-1}{2}\right) \leq \tau\left(\frac{l_0+l_0-1}{2}\right) = \tau(l_0 - \tfrac{1}{2}) \\ < \tau(0.083) < 3.4 = 1.7\tau(1) \leq 1.7\tau(l(y)) < (4/\tau(\tfrac{1}{2}))\tau(|1 - z|).$$

So we may further assume that $y \in [0, 0.3)$. After these reductions it remains to prove the following two lemmas:

Lemma 3.1. *For $y \in [0, 0.3)$, let*

$$(3.3) \quad f(y) = \frac{p(\frac{1}{2} + iy)}{\tau(|\frac{1}{2} - iy|)}.$$

Then f is a strictly decreasing function.

Lemma 3.2. *For $x \in [\frac{1}{2}, 1)$, let*

$$(3.4) \quad g(x) = \frac{p(x)}{\tau(1 - x)}.$$

Then g is a strictly decreasing function.

Proof of Lemma 3.1. By a theorem of Kuz'mina [Ku, p. 198], for all $y \geq 0$, $p(\frac{1}{2} + iy)$ is the capacity of Mori's ring and hence

$$(3.5) \quad p(\frac{1}{2} + iy) = \frac{2\pi}{\mu(s)},$$

where $s = s(y) = ((1 - y/l)/2)^{1/2}$ and $l = l(y) = (y^2 + \frac{1}{4})^{1/2}$. We also express $\tau(|\frac{1}{2} - iy|)$ in terms of the function μ using (2.1):

$$\tau(|\frac{1}{2} - iy|) = \frac{\pi}{\mu(r)},$$

where $r = r(y) = 1/\sqrt{1 + l}$. So we have to prove that the function $f(y) = 2\mu(r)/\mu(s)$ is strictly decreasing for $y \in [0, 0.3)$. We will prove that $df/dy < 0$. Note that

$$(3.6) \quad \frac{dl}{dy} = \frac{y}{l}, \quad \frac{ds}{dy} = -\frac{1}{16sl^3}, \quad \frac{dr}{dy} = -\frac{y}{2l(1 + l)^{3/2}}.$$

Note also that

$$(3.7) \quad s \leq r \quad \text{for all } y \in [0, 0.3).$$

[Proof of (3.7).] $s \leq r$ iff $1 - y/l \leq 2/(l + 1)$ iff $y^2 + \frac{1}{4} - y \leq (1 + y)\sqrt{y^2 + \frac{1}{4}}$ iff $4y^3 - y^2/4 + y + \frac{3}{16} \geq 0$. The last inequality can be easily verified by calculus.]

Now we differentiate f using (2.4) and after some calculations we see that $df/dy < 0$ for $y \in (0, 0.3)$ if and only if

$$(3.8) \quad 8yl^2s^2(s')^2\mathcal{K}(s)\mathcal{K}'(s) < (1 + l)(r')^2\mathcal{K}(r)\mathcal{K}'(r).$$

To prove (3.8) we use Theorem 3.30(2) of [AVV] which says that the function

$$\frac{(t')^2\mathcal{K}(t)\mathcal{K}'(t)}{\log(1/t)}$$

is strictly increasing for $t \in (0, 1)$. Using this fact and (3.7) we see that it suffices to show that

$$(3.9) \quad 8(\log 1/s)yl^2s^2 < (\log 1/r)(1+l), \quad y \in (0, 0.3).$$

We substitute s and r with their expression in terms of y and l and after some calculations (3.9) becomes

$$(3.10) \quad 4y^3 \left(\log \frac{2l}{l-y} \right) + y \log(2l) + y \left(\log \frac{1}{l-y} \right) < 4y^2 \left(\log \frac{2l}{l-y} \right) + \log(1+l) + l \log(1+l).$$

It is clear that

$$4y^3 \left(\log \frac{2l}{l-y} \right) < 4y^2 \left(\log \frac{2l}{l-y} \right)$$

and

$$y \log(2l) < l \log(1+l).$$

So it remains to prove the inequality

$$y \left(\log \frac{1}{l-y} \right) < \log(1+l), \quad y \in (0, 0.3), \quad l = \sqrt{y^2 + \frac{1}{4}}.$$

This can be easily proved using calculus. So the proof of Lemma 3.1 is now complete.

Proof of Lemma 3.2. For $x \in [\frac{1}{2}, 1)$, $p(x)$ has an explicit expression in terms of μ (see [SV, (3.21)]):

$$p(x) = \frac{4}{\pi} \mu(\sqrt{x}) + \frac{\pi}{\mu(\sqrt{x})}.$$

We also write $\tau(1-x)$ in terms of μ using (2.1):

$$\tau(1-x) = \frac{\pi}{\mu(1/\sqrt{2-x})}.$$

Hence

$$(3.11) \quad g(x) = \frac{4}{\pi^2} \mu(\sqrt{x}) \mu(u) + \frac{\mu(u)}{\mu(\sqrt{x})},$$

where $u = 1/\sqrt{2-x}$. Note that

$$(3.12) \quad \frac{du}{dx} = \frac{1}{2(2-x)^{3/2}}, \quad u' = \sqrt{\frac{1-x}{2-x}}.$$

It is easy to see that

$$(3.13) \quad \sqrt{\frac{1}{2}} \leq \sqrt{x} \leq u < 1 \quad \text{for all } x \in [\tfrac{1}{2}, 1)$$

We differentiate g using (3.12) and (2.3) to obtain

$$(3.14) \quad \frac{dg}{dx} = \frac{-\pi \mathcal{K}'(u)}{4x(1-x)\mathcal{K}(\sqrt{x})^2\mathcal{K}(u)} + \frac{-\pi \mathcal{K}'(\sqrt{x})}{4(1-x)\mathcal{K}(\sqrt{x})\mathcal{K}(u)^2} \\ + \frac{-\pi \mathcal{K}(\sqrt{x})}{4(1-x)\mathcal{K}'(\sqrt{x})\mathcal{K}(u)^2} + \frac{\pi \mathcal{K}'(u)}{4x(1-x)\mathcal{K}'(\sqrt{x})^2\mathcal{K}(u)}.$$

Hence $dg/dx < 0$, for $x \in (\frac{1}{2}, 1)$ if and only if

$$(3.15) \quad \mathcal{K}'(u)\mathcal{K}(u)\mathcal{K}(\sqrt{x})^2 \leq \mathcal{K}'(u)\mathcal{K}(u)\mathcal{K}'(\sqrt{x})^2 \\ + x\mathcal{K}(\sqrt{x})\mathcal{K}'(\sqrt{x})^3 + x\mathcal{K}'(\sqrt{x})\mathcal{K}(\sqrt{x})^3$$

or, equivalently,

$$(3.16) \quad x[\mathcal{K}(\sqrt{x})^2 + \mathcal{K}'(\sqrt{x})^2]F(\sqrt{x}) \geq [\mathcal{K}(\sqrt{x})^2 - \mathcal{K}'(\sqrt{x})^2]F(u),$$

where $F(t) = \mathcal{K}(t)\mathcal{K}'(t)$. Now we use Theorem 3.30(1) of [AVV]. It says that the function

$$2(t')^2 F(t)/\pi + \log t$$

is strictly decreasing for $t \in (0, 1)$. Using this fact and (3.13) we see (after some calculations) that in order to prove (3.16) it suffices to prove the following two inequalities:

$$(3.17) \quad \frac{x}{1-x}[\mathcal{K}(\sqrt{x})^2 + \mathcal{K}'(\sqrt{x})^2] \geq \frac{1}{(u')^2}[\mathcal{K}(\sqrt{x})^2 - \mathcal{K}'(\sqrt{x})^2],$$

$$(3.18) \quad \frac{x}{1-x}(-\log \sqrt{x})[\mathcal{K}(\sqrt{x})^2 + \mathcal{K}'(\sqrt{x})^2] \\ \geq \frac{1}{(u')^2}(-\log u)[\mathcal{K}(\sqrt{x})^2 - \mathcal{K}'(\sqrt{x})^2].$$

Note that by (3.13), $\mathcal{K}(\sqrt{x}) > \mathcal{K}'(\sqrt{x})$ and $-\log \sqrt{x} \geq -\log u > 0$. Therefore (3.17) implies (3.18), and thus it remains to prove (3.17). After some calculations (3.17) takes the following form:

$$(3.19) \quad (1-x)\mathcal{K}(\sqrt{x})^2 \leq \mathcal{K}'(\sqrt{x})^2.$$

Now (3.19) follows easily from the inequalities (for $r = \sqrt{x}$):

$$\mathcal{K}(r) \leq \frac{1}{(1-r^2)^{1/2}} \int_0^1 \frac{ds}{(1-s^2)^{1/2}} = \frac{1}{r'} \frac{\pi}{2} \leq \frac{1}{r'} \mathcal{K}'(r).$$

So the proof of Lemma 3.2 is now complete. ■

4. Some Estimates for Conformal Capacity

Let $K = [O, e_1]$ and $L_\theta = [e_\theta, \infty]$, where $e_\theta = (\cos \theta)e_1 + (\sin \theta)e_n$, $\theta \in (0, \pi]$. We will find estimates for the conformal capacity of the ring (K, L_θ) . In the two-dimensional case our main tool is the Green transfinite diameter. In higher dimensions we use polarization.

Fig. 1. The condenser (K, L_θ) .

4.1. Estimates in the Plane

We identify $\mathbf{R}^2$ with $\mathbf{C}$ and use complex notation.

Theorem 4.1. *Let $K = [0, 1]$ and $L_\theta = [e^{i\theta}, \infty]$. Then*

$$(4.1) \quad \tau \left(2 \sin \frac{\theta}{2} \right) \leq \text{cap}(K, L_\theta) \leq \tau \left(\sin^2 \frac{\theta}{2} \right) \leq 2\tau \left(2 \sin \frac{\theta}{2} \right), \quad \theta \in \left(0, \frac{\pi}{3} \right],$$

$$(4.2) \quad 2 = \tau(1) \leq \text{cap}(K, L_\theta) \leq \tau\left(\frac{1}{4}\right) \approx 2.72, \quad \theta \in \left[\frac{\pi}{3}, \frac{\pi}{2} \right),$$

$$(4.3) \quad 2 = \tau(1) \leq \text{cap}(K, L_\theta) \leq \tau(0.618) \approx 2.23, \quad \theta \in \left[\frac{\pi}{2}, \pi \right].$$

Proof. The lower bounds for $\text{cap}(K, L_\theta)$ come easily from circular symmetrization. Below we prove the upper bounds.

Proof of (4.1). Let $L_\theta^* = [1 + \sin^2 \theta/2, \infty]$. We will prove that

$$(4.4) \quad \text{cap}(K, L_\theta) \leq \text{cap}(K, L_\theta^*).$$

Before proving (4.4) we show that it implies (4.1): (K, L_θ^*) is a translation of Teichmüller's ring. Hence

$$\text{cap}(K, L_\theta^*) = \tau \left(\sin^2 \frac{\theta}{2} \right).$$

The last inequality of (4.1) follows easily from the duplication formula (2.2) for τ . Therefore it remains to prove (4.4).

We consider a 1–1 correspondence $j: L_\theta \rightarrow L_\theta^*$, defined as follows: For $s \geq 0$, let $z = (1 + s)e^{i\theta} \in L_\theta$. Then

$$(4.5) \quad \tilde{z} = j(z) = 1 + s + \sin^2 \frac{\theta}{2} \in L_\theta^*.$$

We will show that

$$(4.6) \quad \lambda(\tilde{z}) \geq \lambda(z), \quad z \in L_\theta,$$

where λ denotes the hyperbolic density of the simply connected domain $D = \widehat{\mathbf{C}} \setminus K$.

Let $\hat{z}$ denote the elliptic projection of $z \in L_\theta$ on the positive real axis $\mathbf{R}^+$, i.e., $\hat{z}$ lies on $\mathbf{R}^+$ and satisfies the geometric condition

$$(4.7) \quad |z| + |1 - z| = \hat{z} + (1 + \hat{z}).$$

The principle of the hyperbolic metric [Ha, p. 683] and Joukowski's conformal map (see, e.g., [AVV, p. 129]) show that $\lambda(z) \leq \lambda(\hat{z})$. Hence by the monotonicity of λ on $(1, \infty)$, we see that in order to prove (4.6) it suffices to show $\tilde{z} \leq \hat{z}$. This inequality can be easily proved by a simple calculation based on (4.5) and (4.7). So (4.6) is proved.

Now take any pair of points $z_1, z_2 \in L_\theta$ and denote by $\rho(z_1, z_2)$ their hyperbolic distance in D . By a change of variable, (4.6), and the fact that the interval $(1, \infty)$ is a hyperbolic geodesic of D , we have

$$(4.8) \quad \begin{aligned} \rho(z_1, z_2) &\leq \int_{[z_1, z_2]} \lambda(z) |dz| = \int_{[\tilde{z}_1, \tilde{z}_2]} \lambda(j^{-1}(\tilde{z})) \frac{|dz|}{|d\tilde{z}|} |d\tilde{z}| \\ &\leq \int_{[\tilde{z}_1, \tilde{z}_2]} \lambda(\tilde{z}) |d\tilde{z}| = \rho(\tilde{z}_1, \tilde{z}_2). \end{aligned}$$

Let $T(L_\theta)$ denote the Green transfinite diameter of L_θ with respect to D . Then by (2.10), $\text{cap}(K, L_\theta) = 2\pi T(L_\theta)$. By the results in Subsections 2.2 and 2.3, and by (4.8) we conclude that (4.4) is true. So the proof of (4.1) is complete. ■

Proof of (4.2). By a polarization argument, $\text{cap}(K, L_\theta)$ is a decreasing function of θ . Therefore, by (4.1), for $\theta \in [\pi/3, \pi/2]$ we have

$$\text{cap}(K, L_\theta) \leq \text{cap}(K, L_{\pi/3}) \leq \tau(\sin^2(\pi/6)) = \tau(0.25) \approx 2.72.$$

Proof of (4.3). For $\theta \in [\pi/2, \pi]$ we have as above

$$(4.9) \quad \text{cap}(K, L_\theta) \leq \text{cap}(K, L_{\pi/2}).$$

Note that $L_{\pi/2} = [i, \infty]$. Let $L' = [\frac{1}{2} + i, \infty]$. By a polarization argument

$$(4.10) \quad \text{cap}(K, L_{\pi/2}) \leq \text{cap}(K, L').$$

Using the Joukowski mapping we can easily compute the conformal capacity of the ring (K, L') :

$$(4.11) \quad \text{cap}(K, L') = \gamma(2 + \sqrt{5}) \approx \gamma(4.236) \approx \tau(0.6181) \approx 2.23.$$

In the above calculation we used the definition of the capacity γ of Grötzsch's ring and (2.5). Now (4.3) follows from (4.9), (4.10), and (4.11). The proof of the theorem is now complete. ■

Table 1

θ	L.B. 1	L.B. 2	$p(e^{i\theta})$	$\text{cap}(K, L_\theta)$	Upper bound
$\pi/20$	2.9912	3.1851	3.2831	3.4978	5.0077
$\pi/10$	2.5937	2.8465	2.9016	3.0663	4.1348
$\pi/5$	2.2315	2.5375	2.5451	2.652	3.2896
$\pi/4$	2.1258	2.4363	2.4385	2.526	3.0322
$\pi/3$	2	2.3094	2.3094	2.3721	2.7201
$7\pi/18$	2	2.2446	2.2451	2.2951	2.7201
$8\pi/18$	2	2.1913	2.1928	2.2323	2.7201
$\pi/2$	2	2.147	2.1499	2.1806	2.2315
$2\pi/3$	2	2.0553	2.0610	2.0735	2.2315
π	2	2	2	2	2.2315

Table 1 presents numerical approximations of some quantities associated with the values of θ in the first column. The second and third columns present lower bounds for Teichmüller's function $p(e^{i\theta})$. "L.B.1" is obtained by circular symmetrization, while "L.B.2" is obtained via the method involving the duplication formula for p (see Subsection 2.1). In the fourth column we have values of $p(e^{i\theta})$. These values have been computed by using the software of [AVV]. The computation is based on the explicit expression for the function p in terms of complete elliptic integrals of complex argument [Ku, p. 192]. The numerical approximations of $\text{cap}(K, L_\theta)$ in the fifth column have been found by C. Hu who used his numerical program described in [Hu]. This program solves the problem of the accessory parameters in the Schwarz–Christoffel transformation for doubly connected polygonal domains. Hu's work is closely related with earlier results of H. Daepfen, T. A. Driscoll, and L. N. Trefethen. We refer to [TD] for a survey on numerical Schwarz–Christoffel mappings. Finally, in the sixth column of Table 1 we have the upper bounds for $\text{cap}(K, L_\theta)$ that come from Theorem 4.1. Hu's program is written in *FORTAN*. The other computations have been made by *Mathematica* and *MATLAB*.

4.2. Estimates in Space

We now prove some estimates for conformal capacity in higher dimensions.

Theorem 4.2. Suppose that $n \geq 3$, $K = [O, e_1]$, and $L_\theta = [e_\theta, \infty]$. Then

$$(4.12) \quad \tau_n\left(2 \sin \frac{\theta}{2}\right) \leq \text{cap}(K, L_\theta) \leq 2\tau_n(\sin \theta) \leq 2.15\tau_n\left(2 \sin \frac{\theta}{2}\right), \quad \theta \in \left(0, \frac{\pi}{3}\right],$$

$$(4.13) \quad \tau_n(1) \leq \text{cap}(K, L_\theta) \leq 2\tau_n(\sin \theta) \leq 2.15\tau_n(1), \quad \theta \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right],$$

$$(4.14) \quad \tau_n(1) \leq \text{cap}(K, L_\theta) \leq 2\tau_n(1), \quad \theta \in \left[\frac{\pi}{2}, \pi\right].$$

Proof. The lower bounds come from spherical symmetrization. To prove the upper bounds, suppose first that $\theta \in (0, \pi/2]$. Let $d = \sin \theta = \text{dist}(e_\theta, K)$. Let $L' = [e_\theta, (1 +$

$d)e_\theta]$ and $L'' = [(1+d)e_\theta, \infty]$. An application of Theorem 2.1 (polarization with respect to the sphere $\{x: |x - (1+d)e_\theta| = d\}$) gives

$$(4.15) \quad \text{cap}(K, L') \leq \text{cap}(K, L'').$$

Another polarization argument (or, alternatively, Lemma 5.27 in [V1]) gives

$$(4.16) \quad \text{cap}(K, L'') \leq \tau_n(d) = \tau_n(\sin \theta).$$

Since $L_\theta = L' \cup L''$, the subadditivity of conformal capacity [AVV, Chap. 8], (4.15), and (4.16) imply

$$(4.17) \quad \text{cap}(K, L_\theta) \leq \text{cap}(K, L') + \text{cap}(K, L'') \leq 2\tau_n(\sin \theta), \quad \theta \in (0, \pi/2].$$

Now, if $\theta \in (0, \pi/3]$, then

$$\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \geq 2 \sin \frac{\theta}{2} \cos \frac{\pi}{6} = \sqrt{3} \sin \frac{\theta}{2}.$$

Hence (4.17) and (2.4) imply

$$\text{cap}(K, L_\theta) \leq 2\tau_n(\sin \theta) \leq 2\sqrt{\frac{2}{\sqrt{3}}} \tau_n\left(2 \sin \frac{\theta}{2}\right) \leq 2.15\tau_n\left(2 \sin \frac{\theta}{2}\right)$$

and (4.12) is proved.

If $\theta \in [\pi/3, \pi/2]$, then by (4.12):

$$\text{cap}(K, L_\theta) \leq \text{cap}(K, L_{\pi/3}) \leq 2.15\tau_n\left(2 \sin \frac{\pi}{6}\right) = 2.15\tau_n(1)$$

and (4.13) is proved.

Finally, if $\theta \in [\pi/2, \pi]$, then again by (4.12):

$$\text{cap}(K, L_\theta) \leq \text{cap}(K, L_{\pi/2}) \leq 2\tau_n\left(\sin \frac{\pi}{2}\right) = 2\tau_n(1)$$

and (4.14) is proved.

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